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Conservation Laws and Associated Noether Type Vector Fields via Partial Lagrangians and Noether's Theorem for the Liang Equation

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Abstract We show how one can construct conservation laws of the Liang equation which is not variational but may be regarded as Euler-Lagrange in *part*. This first requires the determination of the Noether-type symmetries associated with the partial Lagrangian. The final construction of the conservation laws resort to a formula equivalent to Noether's theorem. A variety of subclasses are given and, for each, a large number of conserved flows are found—the method is usable for any general choice of the variable speed of sound.

Keywords Inhomogeneous wave and Liang equation · Noether-type symmetries · Partial Lagrangians · Conservation laws

1 Introduction

A systematic and, by now a well known, way of determining conservation laws for systems of Euler-Lagrange equations once their Noether symmetries are known is via Noether theorem. This theorem relies on the availability of a Lagrangian and the corresponding Noether symmetries which leave invariant the action integral.

The Liang equation [7] describes the one-dimensional isentropic motion of a relativistic fluid and is equivalent to a wave equation with non constant coefficients in an inhomogeneous medium, viz.,

$$u_{tt} - s^2(x)u_{xx} - \left(1 - \frac{s^2(x)}{c^2}\right)u_x = 0, \tag{1}$$

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where $s(x)$ is the speed of sound dependent on the equation of state of the fluid and c is a constant being the speed of light. The equation has also been derived by Landau and Lifchitz [6] in the study of gas dynamics. The actual role of the variables u, x and t in the context of relativistic fluids is done in detail by Carbonaro and Giambo [2]. However, the equation, in its usual interpretations of the variables (t and x representing time and space, respectively) arises in a number of situations which is why it has been studied widely and analysed through a number of methods with variations on the inhomogeneous part, for e.g., being a function of u only as in the Gordon equations. In Carbonaro and Giambo [2], the Lie point symmetry generators of (2) are obtained for various classes of $s(x)$ and reductions obtained as a result.

Of interest here are the possible conservation laws of (2) which have a number of well known and well documented physical and theoretical implications for the equation like physical conservation laws related to symmetry, reductions of the equation and potential symmetry generators (see [1, 3, 4, 10]). The most convenient way of determining conservation laws is via Noether's theorem [9] but this is only of use in the variational case. In the absence of a Lagrangian, which is the case for (2), one has to resort to a number of adhoc methods to determine these. However, we use a recently developed novel method of 'partial Lagrangian' and 'Noether type generators' [5, 8] which enjoys the convenience of the method used in the variational case and a formula equivalent to Noether's theorem to determine exact conservation laws. We, consequently, obtain a number of interesting conserved flows for (2) associated with Noether type generators that may appear to be of typical physical value like conservation of energy, spin and those coming from scaling generators.

We present a discussion on the preliminaries, definitions and method used. Suppose (t, x) and (u, v) are the independent and dependent variables, respectively. The *total derivative operator* with respect to t is

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tt}\frac{\partial}{\partial v_t} + v_{tx}\frac{\partial}{\partial v_x} + \cdots$$

and equivalently for D_x. The Euler-Lagrange operators are

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + \cdots, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_t^2\frac{\partial}{\partial v_{tt}} + D_x^2\frac{\partial}{\partial v_{xx}} + \cdots. \end{aligned} \tag{2}$$

A *Lie symmetry generator* will be denoted by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial v} \tag{3}$$

where, for a point generator, ξ, τ, ϕ and ζ are functions of (t, x, u, v).

Consider a k-th order differential equation

$$\begin{aligned} E^1(t, x, u, v, u_t, u_x, v_t, v_x, \ldots) = 0, \\ E^2(t, x, u, v, u_t, u_x, v_t, v_x, \ldots) = 0. \end{aligned} \tag{4}$$

A *conserved flow* of (4) (T^1, T^2) is a vector along which the conservation law

$$D_tT^1 + D_xT^2 = 0 \tag{5}$$

is satisfied along the solutions of (4). If there exists a function $L(t,x,u,v,u_t,u_x,v_t,v_x,\dots)$ such that

$$\frac{\delta L}{\delta u}=0, \qquad \frac{\delta L}{\delta v}=0 \tag{6}$$

satisfies (4), we say (4) is variational and L is a Lagrangian of (4). If (6) does not satisfy (4) completely but

$$\frac{\delta L}{\delta u}=E_0^1, \qquad \frac{\delta L}{\delta v}=E_0^2 \tag{7}$$

where $E^1=E_0^1+\cdots$ and $E^2=E_0^2+\cdots$, we say L is a *partial Lagrangian* of (4). A generator of the type X in (3) is a *Noether type symmetry* corresponding to a partial Lagrangian L if it satisfies

$$XL+L(D_t\tau+D_x\xi)=W^1\frac{\delta L}{\delta u}+W^2\frac{\delta L}{\delta v}+D_tf+D_xg \tag{8}$$

for some gauge vector (f,g), $W^1=\phi-u_t\tau-u_x\xi$, $W^2=\zeta-v_t\tau-v_x\xi$ and X prolonged accordingly.

Note: For Euler-Lagrange equations, (8) reduces to $XL+L(D_t\tau+D_x\xi)=D_tf+D_xg$ and X is a Noether symmetry which leaves invariant the action integral and is also a Lie symmetry of the Euler-Lagrange equations. For partial Lagrangians, the Noether type symmetries need not be symmetries of the differential equation (see [5]).

Corresponding to each Noether type symmetry X of partial Lagrangian L of first-order, there exists a conserved flow (T^1,T^2) of the system (4) given by

$$\begin{aligned}T^1&=L\tau+W^1\frac{\partial L}{\partial u_t}+W^2\frac{\partial L}{\partial v_t}-f,\\ T^2&=L\xi+W^1\frac{\partial L}{\partial u_x}+W^2\frac{\partial L}{\partial v_x}-g.\end{aligned} \tag{9}$$

For second-order Lagrangians, equation (9) becomes

$$\begin{aligned}T^1&=L\tau+W^1\frac{\partial L}{\partial u_t}+W^2\frac{\partial L}{\partial v_t}+[D_jW^\alpha-W^\alpha D_j]\frac{\partial L}{\partial u^\alpha_{tj}}-f,\\ T^2&=L\xi+W^1\frac{\partial L}{\partial u_x}+W^2\frac{\partial L}{\partial v_x}+[D_jW^\alpha-W^\alpha D_j]\frac{\partial L}{\partial u^\alpha_{xj}}-g,\end{aligned} \tag{10}$$

where $u^1=u$ and $u^2=v$. These 'formulae' for the conserved flow are the same as those in Noether's theorem even though the generator X is not a Lie symmetry of the equations (4).

2 Conservation Laws

A partial Lagrangian that may be applicable to (1) is

$$L=\frac{1}{2}u_t^2-\frac{1}{2}s^2(x)u_x^2 \tag{11}$$

so that

$$\frac{\delta L}{\delta u}=\left(2ss'-1+\frac{s^2}{c^2}\right)u_x. \tag{12}$$

Putting these into (8) for $X=a(x,t,u)\partial_x+b(x,t,u)\partial_t+d(x,t,u)\partial_u$ with corresponding gauge vector (f,g) yields

$$\begin{aligned}
&d_t u_t - b_t u_t^2 + d_u u_t^2 - s^2 d_x u_x - a_t u_t u_x\\
&\quad + s^2 b_x u_t u_x + s^2 a_x u_x^2 - s^2 d_u u_x^2 - a s s_x u_x^2 + \frac{1}{2}\big(u_t^2 - s^2 u_x^2\big)(b_t + a_x)\\
&\quad - (f_t + u_t f_u + g_x + u_x g_u) - (d - u_t b - u_x a)\left(2 s s_x - 1 + \frac{s^2}{c^2}\right) u_x = 0. \qquad (13)
\end{aligned}$$

Separating by monomials, we have the determining equations

$$\begin{array}{ll}
u_x^3: & a_u = 0,\\
u_t^3: & b_u = 0,\\
u_x^2: & -a + \dfrac{s^2}{c^2}a + \dfrac{1}{2}s^2 a_x - \dfrac{1}{2}s^2 b_t - s^2 d_u + s s' a = 0,\\
u_t^2: & \dfrac{1}{2}a_x - \dfrac{1}{2}b_t + d_u = 0,\\
u_x u_t: & -a_t + s^2 b_x + b\left(-1 + \dfrac{s^2}{c^2} + 2 s s'\right) = 0,\\
u_x: & -s^2 d_x - g_u - d\left(-1 + \dfrac{s^2}{c^2} + 2 s s'\right) = 0,\\
u_t: & d_t - f_u = 0,\\
: & f_t + g_x = 0.
\end{array} \qquad (14)$$

Solving these usually involves adhoc techniques and even though the system is linear, they may prove to be tedious and cumbersome. Also, $s(x)$ is a general function and the calculations can proceed to completion by specific forms of s otherwise only a subclass of solutions for the vector field X are obtained, if at all.

2.1 Case (i)

In the first case, we suppose particular forms of the vector field X and construct associated conservation laws. This would lead to particular corresponding forms of $s(x)$.

As a first case, suppose the 't translation' generator, $X=\partial_t$ usually associated with conservation of energy. From (14), we get $2ss' + \frac{s^2}{c^2} = 1$ so that $s(x) = \pm\sqrt{c^2 - Ae^{-\frac{x}{c^2}}}$, A constant (note that $s \to \pm c$ as $x \to \infty$). Also, $f_u = g_u = 0$. By (8), the conserved density and flux are, respectively,

$$T^1 = L - u_t^2 - f, \qquad T^2 = s^2 u_t u_x - g$$

so that the conservation law is

$$D_t T^1 + D_x T^2 = -u_t\left[u_{tt} - s^2(x) u_{xx} - \left(1 - \frac{s^2(x)}{c^2}\right) u_x\right]$$

yielding the usual 'multiplier' $W=-u_t$ which significantly arises in the variational case.

The generator for Lorentz rotation $t\partial_x + x\partial_t$ leads to, from (14),

$$-1 + \frac{s^2}{c^2} + s s' = 0, \qquad -1 + s^2 + x\left(-1 + \frac{s^2}{c^2} + 2 s s'\right) = 0 \qquad (15)$$

so that $ss' = 1 - \frac{s^2}{c^2}$ and $s(x) = \pm\sqrt{\frac{x-1}{c^2-x}}$ (note that $s \to \pm ic$ as $x \to \infty$). Here, too, $f_u = g_u = 0$ and the conserved flow is given by

$$T^1 = Lx + (-xu_t - tu_x)u_t - f, \qquad T^2 = Lt + (-xu_t - tu_x)(-s^2u_x) - g$$

so that, with the condition (15), the conservation law is

$$D_tT^1 + D_xT^2 = (-xu_t - tu_x)\left[u_{tt} - s^2(x)u_{xx} - \left(1 - \frac{s^2(x)}{c^2}\right)u_x\right]$$

noting the multiplier $W = -xu_t - tu_x$.

Similarly, the dilation vector field in (x,t), $X = t\partial_t + x\partial_x$, yields

$$-1 + s^2 + x\left(-1 + \frac{s^2}{c^2} + 2ss'\right) = 0, \qquad -x\left(1 - \frac{s^2}{c^2} - ss'\right) = 0$$

so that $\frac{s}{1-s^2}\mathrm{d}s = \frac{\mathrm{d}x}{x}$ and $s(x) = \pm\frac{1}{x}\sqrt{x^2 - k}$ ($s \to \pm 1$ as $x \to \infty$) from which we get

$$T^1 = Lt + (-tu_t - xu_x)u_t - f, \qquad T^2 = Lx + (-tu_t - xu_x)(-s^2u_x) - g.$$

2.2 Case (ii)

We now consider choices of $s(x)$ in (14) and determine the forms of Noether type generators X. It turns out that the we obtain a large class of generators whose form is dependent on a number of conditions including equation (2) itself. This allows us some freedom to structure the form of a generator (ansatz) and then construct the associated conserved flow. For e.g., if we assume a and b constant, the calculations lead to both being zero. In this case, we get the following.

(a) If $s(x)$ is a constant $s \neq c$, say, we obtain $X = -\mathcal{C}^4_x\partial_u$ ($\mathcal{C} = \mathcal{C}(x,t)$, $f = -\mathcal{C}^4_{xt}u$, $g = \mathcal{C}^4_{tt}u - \frac{1}{c^2}\mathcal{C}^7u$ subject to

$$c^2\mathcal{C}^4_{tt} - c^2s^2\mathcal{C}^4_{xx} + c^2\mathcal{C}^4_x - s^2\mathcal{C}^4_x - \mathcal{C}^7 = 0. \tag{16}$$

Thus, the conserved flow is given by

$$T^1 = -\mathcal{C}^4_xu_t + \mathcal{C}^4_{xt}u, \qquad T^2 = s^2\mathcal{C}^4_xu_x - \left(\mathcal{C}^4_{tt}u - \frac{1}{c^2}\mathcal{C}^7u\right)$$

so that

$$\begin{aligned} D_tT^1 &= -\mathcal{C}^4_{xt}u_t - \mathcal{C}^4_xu_{tt} + \mathcal{C}^4_{xtt}u + \mathcal{C}^4_{xt}u_t, \\ D_xT^2 &= s^2\mathcal{C}^4_{xx}u_x + s^2\mathcal{C}^4_xu_{xx} - \left(\mathcal{C}^4_{xtt}u + \mathcal{C}^4_{tt}u_x - \frac{1}{c^2}\mathcal{C}^7_xu - \frac{1}{c^2}\mathcal{C}^7u_x\right) \end{aligned}$$

and the conservation law is

$$D_tT^1 + D_xT^2 = -\mathcal{C}^4_x\left[u_{tt} - s^2(x)u_{xx} - \left(1 - \frac{s^2(x)}{c^2}\right)u_x\right]$$

once the condition (16) is noted in the coefficient of u_x and $\mathcal{C}^7_x = 0$. Once again, the multiplier $W = -\mathcal{C}^4_x$ appears as in the variational case.

Another set of conservation laws for this case arises from the Noether type generators of the form with $a = \mathcal{C}e^{x(\frac{1}{s^2}-\frac{1}{s^2})}$, $b = 0$, $d = \frac{1}{2c^2s^2e^{\frac{x}{c^2}}}(-2e^{\frac{x}{c^2}}\mathcal{C}_x^9c^2s^2 - e^{\frac{x}{s^2}}c^2\mathcal{C}u + e^{\frac{x}{s^2}}s^2\mathcal{C}u)$, $f = -\mathcal{C}_{xt}^9u$ and $g = \frac{1}{c^2}(\mathcal{C}_{tt}^9c^2u + \mathcal{C}^{12}u)$ where $\mathcal{C}_{tt}^9c^2 - \mathcal{C}_{xx}^9c^2s^2 + \mathcal{C}_x^9c^2 - \mathcal{C}_x^9s^2 + \mathcal{C}^{12} = 0$ and $\mathcal{C}$ is a constant. For e.g., by $X = e^{x(\frac{1}{s^2}-\frac{1}{s^2})}\partial_x + \frac{1}{2c^2s^2e^{\frac{x}{c^2}}}(-e^{\frac{x}{s^2}}c^2u + e^{\frac{x}{s^2}}s^2u)\partial_u$ with $f = g = 0$ yields the conserved vectors

$$T^1 = (d - au_x)u_t, \qquad T^2 = La + (d - au_x)(-s^2u_x).$$

(b) For $s = x$, (2) becomes $u_{tt} - x^2(x)u_{xx} - (1 - \frac{x^2}{c^2})u_x = 0$. This choice of s in the homogeneous case is significant as it produces potential (nonlocal) symmetries. As before, (16) yields a cumbersome form for X which we analyse by a manageable choice of the coefficients in X. In particular, if we let a and b be constants, they result in being zero and $X = -\mathcal{C}_x^7(x,t)\partial_u$. We obtain $f = -\mathcal{C}_{xt}^7u$, $g = \mathcal{C}_{tt}^7u - \frac{1}{c^2}\mathcal{C}^{12}u$, $\mathcal{C}_x^{12} = 0$ subject to

$$2c^2\mathcal{C}_{tt}^7 - 2c^2x^2\mathcal{C}_{xx}^7 - 4c^2x\mathcal{C}_x^7 + 2c^2\mathcal{C}_x^7 - 2x^2\mathcal{C}_x^7 - \mathcal{C}^{12} = 0$$

so that the conserved density and flux are, respectively,

$$T^1 = -c_x^7u_t + c_{xt}^7u, \qquad T^2 = x^2c_x^7u_x - \mathcal{C}_{tt}^7u + \frac{1}{c^2}\mathcal{C}^{12}u$$

for which the conservation law is

$$D_tT^1 + D_xT^2 = \mathcal{C}_x^7\left[u_{tt} - x^2(x)u_{xx} - \left(1 - \frac{x^2}{c^2}\right)u_x\right].$$

(c) Similarly, $s = e^{mx}$ give rise to, inter alia, Noether type vector fields $X = \mathcal{C}_x^3\partial_u$, $f = -\mathcal{C}_{xt}^3u$ and $g = -\frac{1}{c^2}\mathcal{C}_{tt}^3c^2u$ subject to $\mathcal{C}_{tt}^3c^2 - e^{2m}c^2x^2\mathcal{C}_{xx}^3 - 2c^2mx^2e^{2m}\mathcal{C}_x^3 - x^2e^{2m}\mathcal{C}_x^3 + c^2\mathcal{C}_x^3 = 0$ with conservation laws found as above.

Concluding Remarks

We have shown that the partial Lagrangian formulation to determine conservation laws is as efficient as the variational one in determining conservation laws from associated vector fields. This has been especially useful here where we have constructed conservation laws for an equation that is not of a Lagrangian formulation, viz., the well known Liang (inhomogeneous wave) equation from, inter alia, relativistic fluid dynamics. Determining the conservation laws for this equation by definition would be a tedious task. A number of classes of the equation, viz., the variation in the choices of the speed of sound $s(x)$, have been enumerated and interesting results obtained. From here, potential symmetries and further classifications may be pursued.

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